4 - 8 Critical points. Linearization.
Find the location and type of all critical points by linearization.

5. $y_1' = y_2$

$y_2' = -y_1 + \frac{1}{2} y_1^2$

```
ClearAll["Global`*"]
```

$\text{Solve}\left[-y_1 + \frac{1}{2} y_1^2 == 0,\ y_1\right]$

$\{\{y_1 \to 0\}, \{y_1 \to 2\}\}$

I will need the information contained in Table 4-1, p. 149 and Table 4-2, p. 150. In fact, because of their importance, I should put them in here, the first 4-1.

Name	$p=\lambda_1+\lambda_2$	$q=\lambda_1\lambda_2$	$\Delta=(\lambda_1-\lambda_2)^2$	Comments on λ_1, λ_2
(a) Node		$q>0$	$\Delta\geq0$	Real, same sign
(b) Saddle point		$q<0$		Real, opposite signs
(c) Center	$p=0$	$q>0$		Pure imaginary
(d) Spiral point	$p\neq0$		$\Delta<0$	Complex, not pure imaginary

followed by 4-2

Type of Stability	$p=\lambda_1+\lambda_2$	$q=\lambda_1\lambda_2$
(a) Stable and attractive	$p<0$	$q>0$
(b) Stable	$p\leq0$	$q>0$
(c) Unstable	$p>0$ OR	OR $q<0$

So there will be two critical points, {0,0}, and {2,0}. First I look at {0,0}, with the eigensystem as

```
Eigensystem[{{0, 1}, {-1, 0}}]
{{i, -i}, {{-i, 1}, {i, 1}}}
```

And manipulating the two eigenvalues to get p, q, and Δ

```
ev1 = i
i

ev2 = -i
-i

p = ev1 + ev2
0
```

```
q = ev1 ev2
1
```

```
Δ = (ev1 - ev2)^2
-4
```

And finding their fate in the grids,

This would be a stable center point. Equals text answer.

For the point (2, 0)

```
Eigensystem[{{0, 1}, {1, 0}}]
{{-1, 1}, {{-1, 1}, {1, 1}}}
```

```
ev1 = -1
-1
```

```
ev2 = 1
1
```

```
p = ev1 + ev2
0
```

```
q = ev1 ev2
-1
```

```
Δ = (ev1 - ev2)^2
4
```

And going to the grids with these,

This would be a unstable saddle point. Equals text answer.

7. $y_1' = -y_1 + y_2 - y_2^2$
$y_2' = -y_1 - y_2$

```
ClearAll["Global`*"]
```

```
Solve[-y1 - y2 == 0, y2]
{{y2 → -y1}}
```

```
Solve[2 y2 - y2^2 == 0, y2]
{{y2 → 0}, {y2 → 2}}
```

This will give the set of points {0,0} and {-2,2},

$\begin{pmatrix} -1 & 1-2\,y2 \\ -1 & -1 \end{pmatrix}$ is the general form of the Jacobian. So starting with the first point {0,0}

```
Eigensystem[{{-1, 1}, {-1, -1}}]
{{-1 + i, -1 - i}, {{-i, 1}, {i, 1}}}

e1 = -1 + i
e2 = -1 - i
-1 + i

-1 - i

p == e1 + e2
p == -2

q == e1 e2
q == 2

Δ = (e1 - e2)^2
-4
```

According to Tables 4-1 and 4-2, the critical point under consideration is a spiral point, and which is stable and attractive. $p = -2,\ q = 2,\ \Delta = -4.$

An interesting implication of the answer is that in finding critical points, the derivatives of all factors count.

Using the Jacobian system for the point $(-2, 2)$

```
Eigensystem[{{-1, -3}, {-1, -1}}]
{{-1 - √3, -1 + √3}, {{√3, 1}, {-√3, 1}}}

ev1 = -1 - √3
-1 - √3

ev2 = -1 + √3
-1 + √3

p = ev1 + ev2
-2

q = ev1 ev2
(-1 - √3) (-1 + √3) // N
-2.
```

```
Δ = (ev1 - ev2)^2
12
```

This would be a saddle point. Equals text answer. The text does not address stability, but 4-2 suggest unstable.

9 - 13 Critical points of ODEs
Find the location and type of all critical points by first converting the ODE to a system and then linearizing it.

9. $y'' - 9y + y^3 = 0$

```
ClearAll["Global`*"]
```

Rearranging

```
y'' = 9 y - y^3
9 y - y^3
```

Making a substitution to get more things to work with

```
y1' = y2
y2
```

I let y_1=y and then

```
y2' = 9 y1 - y1^3
9 y1 - y1^3
```

```
Solve[9 y1 - y1^3 == 0, y1]
{{y1 → -3}, {y1 → 0}, {y1 → 3}}
```

With the y_2 standing by itself above, it will always be zero. So I have three points to consider: {-3,0}, {0,0}, {3,0}.

Stepping in here with the Jacobian system, the prototype matrix is $\begin{pmatrix} 0 & 1 \\ 9-3\,y1^2 & 0 \end{pmatrix}$. So for the point (0, 0),

```
Eigensystem[{{0, 1}, {9, 0}}]
{{-3, 3}, {{-1, 3}, {1, 3}}}
```

Eigenvalues not imaginary, but not equal.

```
e1 = 3
e2 = -3
```

3

-3

```
p = e1 + e2
```

0

```
q = e1 e2
```

-9

$\Delta = (e1 - e2)^2$

36

So for the critical point (0, 0) I have a saddle point by Table 4-1, and it is unstable by Table 4-2.

Again looking at the Jacobian system, the prototype matrix is $\begin{pmatrix} 0 & 1 \\ 9 - 3\,y1^2 & 0 \end{pmatrix}$. So for the point (3, 0),

```
Eigensystem[{{0, 1}, {-18, 0}}]
```

$\{\{3\,i\sqrt{2}, -3\,i\sqrt{2}\}, \{\{-\frac{i}{3\sqrt{2}}, 1\}, \{\frac{i}{3\sqrt{2}}, 1\}\}\}$

$ee1 = 3\,i\sqrt{2}$

$3\,i\sqrt{2}$

$ee2 = -3\,i\sqrt{2}$

$-3\,i\sqrt{2}$

```
p = Simplify[ee1 + ee2]
```

0

```
q = Simplify[ee1 ee2]
```

18

$\Delta = \text{Simplify}\left[(ee1 - ee2)^2\right]$

-72

This would be a center point. Agrees with text. The point (–3, 0) would give the same results, also in agreement with the text. And with q<0 table 4-2 says these are unstable.

11. y'' + Cos[y] = 0

```
ClearAll["Global`*"]
```

This problem is similar to Example 1 in Sec 4.5, where the sol'n is based on small angle formula for $\sin x \approx x$. Looking at the answer, it is seen that a peculiarity of the problem is that (0, 0) is not a critical point, since cos x is not zero there. cos x equals zero at $\frac{\pi}{2}$ and multiples of it.

```
y'' = -Cos[y]
-Cos[y]

y1' = y2
y2

y2' = -Cos[y1]
-Cos[y1]
```

Using the suggestion of the text answer,

$y2' = -\text{Cos}[y1] = -\text{Cos}\left[\pm\frac{\pi}{2} + \tilde{y1}\right] = \text{Sin}\left[\pm\tilde{y1}\right] = \pm\tilde{y1}$

```
y2' = ±ỹ1
±ỹ1
```

What is $\tilde{y1}$? It is a point, something like $\left(\frac{\pi}{2}, 0\right)$. The second value (for y2) will be zero.

$$\text{Eigensystem}\left[\left\{\{0, 1\}, \left\{\frac{\pi}{2}, 0\right\}\right\}\right]$$

$$\left\{\left\{-\sqrt{\frac{\pi}{2}}, \sqrt{\frac{\pi}{2}}\right\}, \left\{\left\{-\sqrt{\frac{2}{\pi}}, 1\right\}, \left\{\sqrt{\frac{2}{\pi}}, 1\right\}\right\}\right\}$$

$$e1 = -\sqrt{\frac{\pi}{2}}$$

$$-\sqrt{\frac{\pi}{2}}$$

$$e2 = \sqrt{\frac{\pi}{2}}$$

$$\sqrt{\frac{\pi}{2}}$$

```
p = e1 + e2
```

0

```
q = e1 e2
```

$-\frac{\pi}{2}$

```
Δ = (e1 - e2)^2
```

2π

So for the point $\tilde{y1} = \left(\frac{\pi}{2}, 0\right)$ I get a saddle point, just as the text said.

$\text{Eigensystem}\left[\left\{\{0, 1\}, \left\{-\frac{\pi}{2}, 0\right\}\right\}\right]$

$\left\{\left\{i\sqrt{\frac{\pi}{2}}, -i\sqrt{\frac{\pi}{2}}\right\}, \left\{\left\{-i\sqrt{\frac{2}{\pi}}, 1\right\}, \left\{i\sqrt{\frac{2}{\pi}}, 1\right\}\right\}\right\}$

$b1 = i\sqrt{\frac{\pi}{2}}$

$i\sqrt{\frac{\pi}{2}}$

$b2 = -i\sqrt{\frac{\pi}{2}}$

$-i\sqrt{\frac{\pi}{2}}$

```
p = b1 + b2
```

0

```
q = b1 b2
```

$\frac{\pi}{2}$

And for the point $\left(-\frac{\pi}{2}, 0\right)$ I get a center, again just as the text predicted.

```
Cos[π/2 + x] == -Sin[x]
```

```
True
```

Checking what seemed reasonable.

```
TableForm[Table[{x, N[Sin[x], 22]},
  {x, -.010000000000, .010000000000, 0.001000000000}], 32]
```

-0.01	-0.00999983
-0.009	-0.00899988
-0.008	-0.00799991
-0.007	-0.00699994
-0.006	-0.00599996
-0.005	-0.00499998
-0.004	-0.00399999
-0.003	-0.003
-0.002	-0.002
-0.001	-0.001
0.	0.
0.001	0.001
0.002	0.002
0.003	0.003
0.004	0.00399999
0.005	0.00499998
0.006	0.00599996
0.007	0.00699994
0.008	0.00799991
0.009	0.00899988
0.01	0.00999983

Below is the answer for sin 0.001 which Mathematica is holding in memory:

```
0.0009999998333333408`
```

This is still a approximation.

13. y'' + Sin[y] = 0

```
ClearAll["Global`*"]
```

This one looks just like the last one.

```
y'' = -Sin[y]
-Sin[y]

y1' = y2
y2

y2' = -Sin[y1]
```

The difference from the last problem may consist in the fact that sin is 0 at (0, 0).

Trying to use the Jacobian approach, $\begin{pmatrix} 0 & 1 \\ -\cos x & 0 \end{pmatrix}$ would be the Jacobian standard matrix, I believe. So for $x = \pm 2 n \pi$, it should be

```
Eigensystem[{{0, 1}, {-1, 0}}]
{{i, -i}, {{-i, 1}, {i, 1}}}

e1 = i
i

e2 = -i
-i

p = e1 + e2
0

q = e1 e2
1

Δ = (e1 - e2)^2
-4
```

This would be a center point, in agreement with the text. For $x = \pi \pm 2 n \pi$,

```
Eigensystem[{{0, 1}, {1, 0}}]
{{-1, 1}, {{-1, 1}, {1, 1}}}
```

This would be a saddle point, in agreement with the text. (p=0, q<0).

15. Trajectories. Write the ODE **y'' - 4 y + y³ = 0** as a system, solve it for y_2 as a function of y_1, and sketch or graph some of the trajectories in the phase plane.

I do not follow the problem's instructions to make a system.

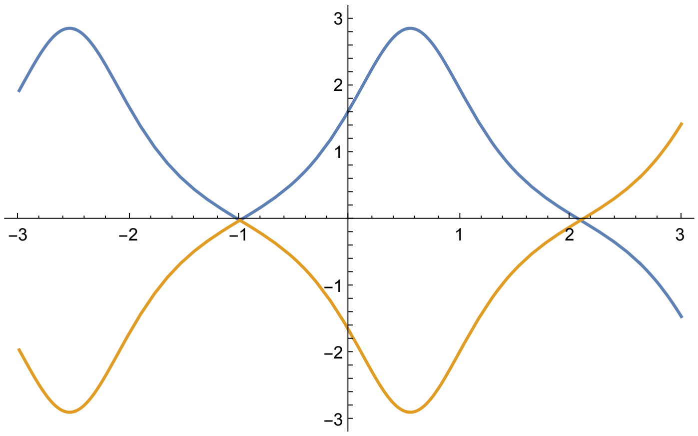
```
eqn = y''[x] - 4 y[x] + y[x]^3 == 0
-4 y[x] + y[x]^3 + y''[x] == 0
```

```
sol = DSolve[eqn, y, x];
```

Solve::ifun:
  Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

The solution is complex, and fairly dense. There is some success in plotting (both halves of) the solution in regular x-y space.

```
Plot[Evaluate[y[x] /. sol /. {C[1] → 1, C[2] → 1}],
 {x, -3, 3}, PlotRange → All]
```



Trying for phase space is not very successful. I am able to show one half of the solution, the ‘negative’ half. But it doesn’t look like the text, or like what I would expect.

$$\textbf{StreamPlot}\Big[\Big\{y,\ -3\,\mathrm{i}\sqrt{\frac{2}{-4+3\sqrt{2}}}\ \textbf{JacobiSN}\Big[\frac{\sqrt{-4-3\sqrt{2}-8x-6\sqrt{2}\,x-4x^2-3\sqrt{2}\,x^2}}{\sqrt{2}},\ \frac{4-3\sqrt{2}}{4+3\sqrt{2}}\Big]+\frac{4\,\mathrm{i}\ \textbf{JacobiSN}\Big[\frac{\sqrt{-4-3\sqrt{2}-8x-6\sqrt{2}\,x-4x^2-3\sqrt{2}\,x^2}}{\sqrt{2}},\ \frac{4-3\sqrt{2}}{4+3\sqrt{2}}\Big]}{\sqrt{-4+3\sqrt{2}}}\Big\},$$

$$\{x,-10,10\},\ \{y,-10,10\},\ \textbf{Frame}\to\textbf{True}\Big]$$